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We study a general macroscopic quantum system of a finite size, which will exhibit a symmetry breaking if the system size goes to infinity, when the system interacts with an environment. We evaluate the decoherence rates of the anomalously fluctuating vacuum (AFV), which is the symmetric ground state, and the pure phase vacua (PPVs). By making full use of the locality and huge degrees of freedom, we show that there can exist an interaction with an environment which makes the decoherence rate of the AFV anomalously fast, whereas PPVs are less fragile.

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We consider a macroscopic quantum system which can exhibit spontaneous symmetry breaking (SSB). According to experience, it is very hard to observe superpositions of states with different values of the order parameter. We call the ground state of such pure states the *anomalously fluctuating vacuum* (AFV) because it has anomalously large fluctuations of macroscopic variables. When the system volume is infinite, the reason for the impossibility of observing the AFV is obvious; there is no local operator intertwining the macroscopically distinct states, and thus their superposition is a mixed state rather than a pure state [1,2]. However, this reasoning cannot be applied to finite systems, and any superposition of pure states is a pure state (except when a superselection rule forbids it) in quantum theory of finite *closed* systems. Hence, the answer should come from the fact that real physical systems are not completely closed; there are interactions with surrounding environments. Effects of environments have been discussed intensively in studies of, e.g., 'macroscopic quantum coherence' [3] and quantum measurement [4]. However, most previous studies on these subjects assumed that the principal systems of interest were describable by a *small number* of collective coordinates, which interact *non-locally* with some specific environment. Although such models might be applicable to systems which have a non-negligible energy gap to excite 'internal coordinates' of the collective coordinates, there are many systems which do not have such an energy gap. Moreover, the results depended strongly on the choices of the coordinates and the form of the nonlocal interactions, so that general conclusions were hard to draw.

In this work, we study a *general* finite system of *huge* degrees of freedom $|\Lambda|$, which interacts with a *general* environment E via a general *local* interaction $H_{\rm int}$. We derive a lower bound γ of the decoherence rates for the AFV and for 'pure phase vacua' (PPVs), which will be defined later, by making full use of the locality: the interaction must be local (Eq. (5) below) and macroscopic variables must be averages over a macroscopic region (Eq. (3)). To express the locality manifestly, we use a local field theory throughout this work. It is shown that there can exist $H_{\rm int}$ which makes γ of the AFV larger than that of PPVs by an anomalously large factor $\mathcal{O}(|\Lambda|)$. We also derive a lower bound $\Delta\gamma$ of the difference of the decoherence rates between the AFV and PPVs, and show that there can exist $H_{\rm int}$ which makes $\Delta\gamma$ anomalously large, proportional to $\mathcal{O}(|\Lambda_{\rm C}|)$, where $\Lambda_{\rm C}$ ($\subseteq \Lambda$) is the 'contact region' in which the principal system interacts with E. These results show that the AFV is 'fragile' (i.e., decoheres at an anomalously fast rate) for large $|\Lambda|$ and $|\Lambda_{\rm C}|$, however small the coupling constant of $H_{\rm int}$ is, whereas PPVs are less fragile.

We first fix the energy scale ΔE of interest. Since it sets a minimum length scale l, we can treat the system as a lattice system Λ whose lattice constant is l. In some cases, the degrees of freedom (the number of lattice cites) $|\Lambda|$ of the effective theory can become small even for a macroscopic system when, e.g., a non-negligible energy gap exists in ΔE, so that the number of quantum states in ΔE is small. Some SQUID systems are such examples. We here exclude such systems, and concentrate on systems whose $|\Lambda|$ is a macroscopic number. Although l is somewhat arbitrary, this ambiguity does not change the conclusions of the present paper. We take $l = 1$, and consider the case where Λ is a d-dimensional hypercubic lattice system L^d. For simplicity, we impose the periodic boundary conditions, and assume that all states under consideration are invariant under the spatial translation. To establish the relation between infinite systems and macroscopic but finite systems of our interest, we consider a sequence of lattice systems $\{\Lambda\}$ with increasing $|\Lambda|$. We assume that it exhibits an SSB as $\Lambda \to \mathbf{Z}^d$, for which a ground state $\Xi_{\mathbf{Z}^d}$ is a pure phase vacuum (PPV) that breaks the symmetry, i.e., the expectation value of some order parameter $m(x)$, which is a one-site observable at site x, is nonvanishing. For a finite $|\Lambda|$, we can find a sequence of pure states $\{\Xi_\Lambda\}$ that approaches $\Xi_{\mathbf{Z}^d}$ as $\Lambda \to \mathbf{Z}^d$ [5–8]. If we take Ξ_Λ as a normalized vector in a Hilbert space on Λ, it satisfies $\nu_\Lambda \equiv (\Xi_\Lambda, m(0)\Xi_\Lambda) \to \nu_{\mathbf{Z}^d} \neq 0$ as $\Lambda \to \mathbf{Z}^d$. Note that $\Xi_{\mathbf{Z}^d}$ has the "cluster property" [1,2], which means that spatial correlations of any local operators vanish at a large distance. To exclude exceptional uninteresting sequences, we require that Ξ_Λ's should also have the cluster property, and call such $\{\Xi_\Lambda\}$, as well as its element Ξ_Λ, a PPV of a finite system. Here, we generalize the notion of the cluster property to finite systems as follows. Since Ξ_Λ is not generally an energy eigenstate, it may evolve with time in finite systems, and we denote Ξ_Λ after an time interval t as $\Xi_\Lambda(t)$. It becomes time-invariant in the limit of $\Lambda \to \mathbf{Z}^d$ because it approaches $\Xi_{\mathbf{Z}^d}$, which is a time-invariant state. Hence, if we introduce a time scale T, which will be taken sufficiently long, then $\nu_{T,\Lambda} := \inf_{0 \le t \le T} |(\Xi_\Lambda(t), m(x)\Xi_\Lambda(t))| \to |\nu_{\mathbf{Z}^d}|$ for any $T > 0$. For a positive number ε (≤ 1), we define an *ε-correlation region* $\Omega_{T,\Lambda}(y,\varepsilon)$ of a quantum state, the expectation value for which is denoted by $\langle \cdots \rangle$, by its complement,

$$\Omega_{T,\Lambda}(y,\varepsilon)^c := \left\{ x \in \Lambda \;\middle|\; |\langle \delta a^*(t)\delta b(t)\rangle| < \varepsilon(\langle \delta a^*(t)\delta a(t)\rangle)^{1/2}(\langle \delta b^*(t)\delta b(t)\rangle)^{1/2}, \forall a \in \mathcal{A}(x), \forall b \in \mathcal{A}(y), 0 \le \forall t \le T \right\}, \quad (1)$$

where $\mathcal{A}(x)$ and $\mathcal{A}(y)$ denote a set of all one-site operators (independent of Λ) at sites x and y, respectively, and $\delta a := a - \langle a \rangle$, $\delta b := b - \langle b \rangle$. We say that a sequence of states (of finite systems) has a *cluster property* iff the correlation region $\Omega_{T,\Lambda}(y,\varepsilon)$ for any positive ε and T does not depend on $|\Lambda|$ for a sufficiently large $|\Lambda|$. Note that the volume $|\Omega_{T,\Lambda}(\varepsilon, y)|$ is independent of y because of the assumed translational invariance of the states. We thus denote it simply $|\Omega_{T,\Lambda}(\varepsilon)|$. From Eq. (1), one can show for any one-site operator $a(x)$, and for $0 \le \forall t \le T$, that

$$(\Xi_\Lambda(t), \delta A_\Lambda^* \delta A_\Lambda \Xi_\Lambda(t)) \le (|\Omega_{T,\Lambda}(\varepsilon)|/|\Lambda| + \varepsilon)\,(\Xi_\Lambda(t), \delta a^*(0)\delta a(0)\Xi_\Lambda(t)), \quad (2)$$

where A_Λ is an intensive operator composed of a;

$$A_\Lambda := \frac{1}{|\Lambda|}\sum_{x\in\Lambda} a(x), \quad (3)$$

and $\delta A_\Lambda := A_\Lambda - \langle A_\Lambda \rangle$. By taking ε small enough, one can see that fluctuations of any intensive operators are negligible for PPVs, in consistent with thermodynamics. In this sense, PPVs are thermodynamically normal. For the order parameter $M_\Lambda := (1/|\Lambda|)\sum_{x\in\Lambda} m(x)$, in particular, $(\Xi_\Lambda, \delta M_\Lambda^* \delta M_\Lambda \Xi_\Lambda) \to 0$ as $|\Lambda| \to \infty$.

For a finite system, in general, there also exists the ground state $\Phi_{0,\Lambda}$ which *preserves* the symmetry, i.e., $(\Phi_{0,\Lambda}, M_\Lambda \Phi_{0,\Lambda}) = 0$ [5–8]. Since this state consists primarily of a superposition of Ξ_Λ's with different values of ν_Λ, it has a large fluctuation of the order parameter [5–8];

$$(\Phi_{0,\Lambda}, \delta M_\Lambda^* \delta M_\Lambda \Phi_{0,\Lambda}) = \mathcal{O}(|\Lambda|^0), \quad (4)$$

which must be contrasted with that of PPVs. Since such a large fluctuation is anomalous in view of thermodynamics, we call the sequence of $\{\Phi_{0,\Lambda}\}$, as well as its element $\Phi_{0,\Lambda}$, the AFV. It was proved that such a state cannot be a pure state in the infinite-volume limit ($|\Lambda| \to \infty$) [1,2]. In contrast to Ξ_Λ, $\Phi_{0,\Lambda}$ does not evolves with time in a closed system because it is an eigenstate of H_Λ.

As an example, we consider a spin system with the simplest Hamiltonian, $H_\Lambda = -J\sum_{<x,y>} s_3(x)s_3(y)$, which possesses a discrete symmetry, the up-down symmetry. The order parameter is $S_{3,\Lambda} := (1/|\Lambda|)\sum_{x\in\Lambda} s_3(x)$. There are two PPVs, $\Xi_{+,\Lambda} := |+++\cdots\rangle$ and $\Xi_{-,\Lambda} := |---\cdots\rangle$, for which $(\Xi_{\pm,\Lambda}, \delta a(0)\delta b(x)\Xi_{\pm,\Lambda}) = 0$ for $x \ne 0$. Hence, $|\Omega_{0,\Lambda}(0)| = 1$ for these states. Since $\Xi_\pm$ are eigenstates of H_Λ in this simple case, $|\Omega_{T,\Lambda}(0)| = |\Omega_{0,\Lambda}(0)| = 1$ for any T. On the other hand, $\Phi_{0,\Lambda} := (\Xi_+ + \Xi_-)/\sqrt{2}$ is an AFV [13], for which $(\Phi_{0,\Lambda}, S_{3,\Lambda}\Phi_{0,\Lambda}) = 0$ and $(\Phi_{0,\Lambda}, S_{3,\Lambda}^* S_{3,\Lambda}\Phi_{0,\Lambda}) = 1$. As another example, we consider a free boson system confined in a uniform box under the periodic boundary conditions. The order parameter is $m(x) := \psi(x)$. The number state of free bosons $|N\rangle$, which is the number state of the lowest ($k = 0$) single-body state, is the AFV because $\langle N|\delta M_\Lambda^* \delta M_\Lambda|N\rangle = N/|\Lambda| = \mathcal{O}(|\Lambda|^0)$ when N is increased in proportion to $|\Lambda|$. On the other hand, the coherent state of free bosons $|\alpha\rangle$, which is the coherent state of the $k = 0$ state, is easily shown to be a PPV. Unlike these trivial examples, it is generally difficult to find PPVs and AFVs of interacting many-body systems, and to confirm the cluster property of PPVs for *any* observables. A successful example is interacting many bosons confined in a uniform box under the periodic boundary conditions [8–10]. It was shown that the 'coherent state of interacting bosons' (CSIB) $|\alpha, G\rangle$ is a PPV [9], which preserves the cluster property over $T = O(|\Lambda|^{1/2})$ [8]. On the other hand, by superposing $|\alpha, G\rangle$'s over the phase of α, one can construct the 'number state of interacting bosons' (NSIB) $|N, G\rangle$, which is the AFV [9]. The fragility of the NSIB and the robustness of the CSIB were shown in Ref. [10], in consistent with the general theorems presented below.

We study robustness, against weak perturbations from a general environment E, of the AFV and PPVs of a general system (which we hereafter call a 'principal system') of size $|\Lambda|$. The Hilbert space of the total system is the product $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\rm E}$ of the individual Hilbert spaces. The Hamiltonian of the total system is composed of three parts, $H_{\rm tot} := H_\Lambda + H_{\rm int} + H_{\rm E}$, where H_Λ and $H_{\rm E}$ denote the Hamiltonians of the principal system and E, respectively, and $H_{\rm int}$ is an interaction between them. The dimensionless constant λ is small: when the principal system couples strongly to a part of an external system, one must include such a part into the principal system, then, after a proper renormalization process, the principal system couples only weakly to the rest of the external system, which we call here the environment. Most previous work on decoherence of macroscopic systems assumed that the principal system could be described by a *small number* of collective coordinates, which interact *non-locally* with some specific environment. As mentioned in the introduction, however, such formulations are inappropriate for general systems. Therefore, we start from a Hamiltonian $H_{\rm tot}$ with *macroscopically large* degrees of freedom, which interact *locally* with many degrees of freedom of E;

$$H_{\rm int} = \lambda \sum_{x \in \Lambda_{\rm C}} a(x) \otimes b(x), \tag{5}$$

where $a(x)$ and $b(x)$ are local operators of the principal system and E, respectively, at the lattice point $x \in \Lambda_{\rm C}$. Here, $\Lambda_{\rm C}$ $(\subseteq \Lambda)$ is a 'contact region' between the principal system and E. Without loss of generality, we assume that $\mathrm{tr}(\sigma b(x)) = 0$: If it is finite it can be absorbed into $H_{\rm tot}$ as a renormalization term. Putting $a_k := |\Lambda|^{-1} \sum_{x \in \Lambda} a(x) \mathrm{e}^{ikx}$and $b_k := \sum_{x \in \Lambda_{\rm C}} b(x) \mathrm{e}^{-ikx}$, for $k \in (2\pi \mathbf{Z}/L)^d$, yields $H_{\rm int} = \sum_k a_k \otimes b_k$ [12].

The density operator of the total system $\rho_{\rm tot}(t)$ evolves according to $H_{\rm tot}$. We are interested in a reduced density operator of the principal system, $\rho(t) := \mathrm{tr}_{\mathcal{H}_{\rm E}}[\rho_{\rm tot}(t)]$. Since we discuss decoherence of an initially pure state, we assume that $\rho_{\rm tot}$ is initially an uncorrelated product; $\rho_{\rm tot}(0) = \rho(0) \otimes \sigma$, where $\rho(0) := |\phi\rangle\langle\phi|$ is a pure state of the principal system, and σ is a time invariant state of E. We are studying two cases: $\rho(0)$ is (a) the AFV, and (b) a PPV. In the former case $\rho(0)$ is an eigenstate of H_Λ, whereas in the latter $\rho(0)$ is a superposition of low-lying eigenstates of H_Λ [5–8]. In either case, the energy spread (the width of distribution over eigenvalues of H_Λ) of $\rho(0)$ is narrow. It is expected that for a sufficiently small λ the energy spread remains small for a short t of interest. If the energy spread is smaller than $\hbar/\tau_c$, where τ_c denotes the correlation time of b of E, we obtain the following Markovian equation;

$$i\hbar \frac{d}{dt}\rho = [H_\Lambda, \rho] + i\frac{\lambda^2}{\hbar} \sum_{k_1} \sum_{k_2} g_{k_1 k_2} (2 a_{k_2} \rho a_{k_1}^* - \{a_{k_1}^* a_{k_2}, \rho\}), \tag{6}$$

where g is a positive matrix defined by the time correlation in E;

$$g_{k_1 k_2} := \frac{1}{2} \int_{-\infty}^{\infty} ds \langle b_{k_1}^* b_{k_2}(s) \rangle. \tag{7}$$

Due to perturbations from E, initially pure states generally evolve into mixed ones. If a pure state rapidly evolves into a mixed state (i.e., decoheres), such a state should be hard to realize and observe. We say such states are 'fragile.' On the other hand, if a pure state does not decohere for a long time, it should be easy to observe. We say such states are 'robust.' Namely, effects of the environment select out particular states as observable ones. Zurek et al. called this mechanism the 'environment-induced superselection rule' in his discussion on quantum measurements [4]. We apply this idea to the present problem of SSB in a finite system. As a measure of purity of a quantum state, we employ the so-called linear entropy $S_{\rm lin}(\rho) := 1 - \mathrm{tr}[\rho^2]$ [11], which vanishes only for pure states. We evaluate $S_{\rm lin}$ as a power series of λ^2, $S_{\rm lin} = S_{\rm lin}^{(0)} + S_{\rm lin}^{(1)} + \cdots$, where $S_{\rm lin}^{(n)} = \mathcal{O}(\lambda^{2n})$, using the standard interaction picture technique. We confirmed that this series converges [14]. Since $S_{\rm lin}^{(0)} = 0$ for $\rho(0) = |\phi\rangle\langle\phi|$, this suggests that $S_{\rm lin}^{(1)}$ would give the dominant contribution under our assumption that λ is small. It is calculated as

$$S_{\rm lin}^{(1)}(\phi, t) = \frac{\lambda^2}{\hbar^2} \int_0^t ds \sum_{k_1 k_2} g_{k_1 k_2} (\phi, \delta a_{k_1}^*(s) \delta a_{k_2}(s) \phi). \tag{8}$$

If ϕ is translational invariant, both spatially and temporally, the rhs is bounded by the fluctuation of an intensive variable $A_\Lambda := (1/|\Lambda|) \sum_{x \in \Lambda} a(x)$, and we obtain
Theorem 1:

$$S_{\rm lin}^{(1)}(\phi, t) \geq (\lambda^2/\hbar^2) g_{00} (\phi, \delta A_\Lambda^* \delta A_\Lambda \phi) t. \tag{9}$$

Since the rhs is proportional to t, we may interpret it divided by t as a *lower bound of the decoherence rate*, which we denote γ. It is proportional to the fluctuation of the intensive variable A composed of $a(x)$ which constitutes $H_{\rm int}$ as Eq. (5).

To apply this theorem, recall that we are considering a theory which effectively describes phenomena in some energy range of interest. The effective theory can be constructed from an elementary dynamics by an appropriate renormalization process. In this process, in general, many terms would be generated in the effective interaction;

$H_{\rm int} = H_{\rm int}^{[1]} + H_{\rm int}^{[2]} + \cdots$, where $H_{\rm int}^{[\ell]} = \lambda^{[\ell]} \sum_{x\in\Lambda_{\rm C}^{[\ell]}} a^{[\ell]}(x) \otimes b^{[\ell]}(x)$ [15]. Hence, it seems rare that $H_{\rm int}$ does not have a term with $a^{[\ell]}(x) = m(x)$, although $\lambda^{[\ell]}$ might be small. If $\lambda^{[\ell]}$ is small, such a term could be neglected if $|\Lambda|$ were small. However, it becomes relevant in the present case of $|\Lambda| \gg 1$, for the following reason. Such a term yields $\gamma^{[\ell]} = (\lambda^{[\ell]2}/\hbar^2) g_{00}^{[\ell]} \times \mathcal{O}(|\Lambda|^0)$ for γ of the AFV. For PPVs, on the other hand, we can see from Eqs. (2) and (9) that $\gamma^{[\ell]} = (\lambda^{[\ell]2}/\hbar^2) g_{00}^{[\ell]} \times \mathcal{O}(1/|\Lambda|)$ for *any* of $\hat{H}_{\rm int}^{[\ell]}$'s. Since $|\Lambda|$ is a macroscopic number, the former is much larger than the latter. Regarding the factor g_{00}, we can estimate its order of magnitude as follows [12]. Let $\Lambda_{\rm E}^{\rm corr}$ be the *correlation region of E*, i.e., the region of x in which $\int_{-\infty}^{\infty} ds \langle b^*(x) b(0,s)\rangle$ is correlated. When $|\Lambda_{\rm E}^{\rm corr}| > |\Lambda_{\rm C}|$, we can roughly estimate that $g_{00} \propto |\Lambda_{\rm C}|^2$ [14]. Hence, Theorem 1 yields $\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}|^2$ for the AFV [16], whereas $\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}|^2/|\Lambda|$ for PPVs [17]. On the other hand, when $|\Lambda_{\rm E}^{\rm corr}| < |\Lambda_{\rm C}|$, we can roughly estimate that $g_{00} \propto |\Lambda_{\rm C}||\Lambda_{\rm E}^{\rm corr}|$. Hence, $\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}||\Lambda_{\rm E}^{\rm corr}|$ for the AFV, whereas $\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}||\Lambda_{\rm E}^{\rm corr}|/|\Lambda|$ for PPVs. In both cases, we find that the AFV is fragile (i.e., decoheres at an anomalously fast rate), however small λ is, if $|\Lambda_{\rm C}|$ $(\le |\Lambda|)$ is large enough. Therefore, we think that AFVs are almost always fragile in real physical systems. This seems to give microscopic foundations of our experience; AFVs are difficult to observe.

How are PPVs? We have already seen that γ is $\mathcal{O}(1/|\Lambda|)$ times smaller for PPVs than for the AFV. Unlike the case of Ref. [10], however, we cannot draw a general conclusion on the robustness of PPVs because γ is a lower bound. To see more details, we now present another theorem. We can prove it for two cases [14]; the breaking of (a) the $\mathbf{Z}_2$ (parity) symmetry, and (b) the $U(1)$ symmetry, under the assumption that $a(x) = m(x)$ in Eq. (5). We here describe an outline of the proof for case (a). In this case, $m(x)$ transforms as $\mathcal{P} m(x) \mathcal{P}^\dagger = -m(x)$ by the parity operation $\mathcal{P}$. Any vector can be decomposed into even- and odd-parity components, Φ_+ and Φ_-, respectively. Since the Hamiltonian commutes with $\mathcal{P}$, they remain in the even- and odd-parity subspaces, respectively, for any t. We therefore denote them $\Phi_+(t)$ and $\Phi_-(t)$, which are assumed to be normalized. Let us consider a PPV which can be decomposed as $\Xi_\Lambda(t) := c_+\Phi_+(t) + c_-\Phi_-(t)$. By operating $\mathcal{P}$, we obtain another PPV, $\Xi'_\Lambda(t) := \mathcal{P}\Xi_\Lambda(t) = c_+\Phi_+(t) - c_-\Phi_-(t)$. Since Ξ_Λ and Ξ'_Λ must become orthogonal to each other when $|\Lambda| \to \infty$, we obtain $c_+, c_- \to 1/\sqrt{2}$ as $|\Lambda| \to \infty$. For a macroscopic time region $0 \le t \le T$, we denote the ε-correlation region of Ξ_Λ by $\Omega_{T,\Lambda}(\varepsilon)$, and let $\nu_{T,\Lambda} := \inf_{0\le t\le T} |(\Xi_\Lambda(t), M_\Lambda \Xi_\Lambda(t))|$. After lengthy calculations, we obtain

$$|c_+|^2 S_{\rm lin}^{(1)}(\Phi_+,t) + |c_-|^2 S_{\rm lin}^{(1)}(\Phi_-,t) - S_{\rm lin}^{(1)}(\Xi_\Lambda,t) \ge \frac{\lambda^2}{\hbar^2} g_{00} \Big\{ \nu_{T,\Lambda}^2 t - \left[\frac{|\Omega_{T,\Lambda}(\varepsilon)|}{|\Lambda|} + \varepsilon\right] \int_0^t ds (\Xi_\Lambda(s), \delta m^*(0)\delta m(0)\Xi_\Lambda(s)) \Big\}.$$

Noting that any local operator hardly intertwine between Ξ_Λ and Ξ'_Λ, we can show that $|c_+|^2 S_{\rm lin}^{(1)}(\Phi_+,t) + |c_-|^2 S_{\rm lin}^{(1)}(\Phi_-,t) \to S_{\rm lin}^{(1)}(\Phi_{0,\Lambda})$ as $\Lambda \to \mathbf{Z}^d$, and that $\nu_{T,\Lambda}^2 \ge (\Phi_{0,\Lambda}, \delta M_\Lambda^* \delta M_\Lambda \Phi_{0,\Lambda}) + \epsilon'_\Lambda$ with $\epsilon'_\Lambda \to 0$. We thus obtain

Theorem 2 : For a fixed contact region $\Lambda_{\rm C}$,

$$S_{\rm lin}^{(1)}(\Phi_{0,\Lambda},t) - S_{\rm lin}^{(1)}(\Xi_\Lambda,t) \ge \frac{\lambda^2}{\hbar^2} g_{00} t (\Phi_{0,\Lambda}, \delta M_\Lambda^* \delta M_\Lambda \Phi_{0,\Lambda}) + \epsilon_\Lambda. \tag{10}$$

where ϵ_Λ *is a small number, approaching* 0 *as* $\Lambda \to \mathbf{Z}^d$.

Since the rhs is proportional to t, we can interpret it divided by t as a lower bound of the *difference* of the decoherence rates, which we denote $\Delta\gamma$. In a manner similar to the estimation of γ, we can roughly estimate that $\Delta\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}|^2$ for $|\Lambda_{\rm E}^{\rm corr}| > |\Lambda_{\rm C}|$, whereas $\Delta\gamma \propto (\lambda^2/\hbar^2)|\Lambda_{\rm C}||\Lambda_{\rm E}^{\rm corr}|$ for $|\Lambda_{\rm E}^{\rm corr}| < |\Lambda_{\rm C}|$. In both cases, we find that $\Delta\gamma$ is large, however small λ is, if $|\Lambda_{\rm C}|$ $(\le |\Lambda|)$ is large enough. This large term originates from the modes with $k = 0$ [12], whereas Theorem 2 indicates that the other modes with $k \neq 0$ give only a negligible difference. Namely, both the AFV and PPVs decohere by the $k \neq 0$ modes, whereas only the AFV decohere anomalously fast by the $k = 0$ modes. In some cases (e.g., when the environment is violent) the former modes might make both the AFV and PPVs decohere quickly. Hence, unlike the case of Ref. [10], we cannot draw a definite conclusion on the robustness of PPVs for general cases. We can, however, definitely say that PPVs are less fragile than the AFV in the sense that $S_{\rm lin}^{(1)}(\Phi_{0,\Lambda},t) - S_{\rm lin}^{(1)}(\Xi_\Lambda,t) \ge 0$, because the rhs of Eq. (10) is positive.

To demonstrate how the theorems are satisfied, we present simple examples, for which we can explicitly calculate $S_{\rm lin}^{(1)}$. For the simple spin system discussed above, one can easily show by putting $m(x) := s_3(x)$ that $a_k \Xi_+ = \delta_{k0}\Xi_+$ and so on, and that $S_{\rm lin}^{(1)}(\Phi_{0,\Lambda},t) = (\lambda^2/\hbar^2) g_{00} t$, $S_{\rm lin}^{(1)}(\Xi_+,t) = 0$. For the free boson system, on the other hand, if we assume $H_{\rm int} := \sum_x (\psi(x)\otimes b(x) + \psi^*(x)\otimes b^*(x))$, we can show that $S_{\rm lin}^{(1)}(|N\rangle,t) = (\lambda^2/\hbar^2)[n_0(g_{00}^+ + g_{00}^-) + \sum_k g_{kk}^-/|\Lambda|]t$, $S_{\rm lin}^{(1)}(|\alpha\rangle,t) = (\lambda^2/\hbar^2)[\sum_k g_{kk}^-/|\Lambda|]t$, where g_{00}^+, g_{kk}^- are constants determined by correlation functions in E.

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[11] Although the linear entropy does not have some good properties of other entropies, it equals to the Renyi entropy, $S_2[\rho] = -\ln(\mathrm{tr}\rho^2)$, to $\mathcal{O}(\lambda^2)$.
[12] Note that b_0 is *not* the spatially uniform component in E, because $b_0 = \sum_{x\in\Lambda_{\rm C}} b(x) \neq \sum_{x\in\Lambda_{\rm E}} b(x)$. Hence, g_{00} is not determined by the uniform component in E.
[13] Since this example is too simple, the ground states (AFVs) are degenerate.
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[15] We expect that by an appropriate renormalization process $H_{\rm tot}$ can be made local in the relevant space-time scale.
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[17] Although Theorem 1 was derived for time invariant states, it can also be applied to PPVs for a sufficiently large $|\Lambda|$, because for which PPVs are time invariant.